ON THE ASYMPTOTIC STABILITY OF THE SOLUTION OF A NONLINEAR PARABOLIC SYSTEM

(OB ASIMPTOTICHESKOI USTOICHIVOSTI RESHENIIA NELINEINOI PARABOLICHESKOI SISTEMY)

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In [1,2] the author investigated the problem on the stability of the solutions of a nonlinear equation of heat conduction.

Here it will be shown how one can use the theory of semigroups of operators to formulate analogous results for a nonlinear parabolic equation of higher order

$$\frac{\partial u}{\partial t} = -Lu(t, x) + f(t, x, u) \tag{1}$$

with vanishing boundary conditions

$$u|_\Gamma = \frac{\partial u}{\partial n}\bigg|_\Gamma = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}}\bigg|_\Gamma = 0, \qquad u(x, 0) = \varphi(x) \tag{2}$$

in the space $L_p(\Omega)$, i.e.

$$\|u\|^p = \int\limits_\Omega |u(t, x)|^p\, dx, \qquad x = (x_1, \ldots, x_n) \tag{3}$$

Here Lu is a uniformly elliptic operator of order $2m$, whose coefficients (depending on x only), as well as the boundary Γ of the bounded region Ω of an n-dimensional space, satisfy certain conditions of smoothness.

Let us assume that the operator L is representable as the sum of positive definite, selfadjoint (in the sense L_2), and skew-symmetric operators. Then, as was shown by Solomiak [3], the operator L will generate a strongly discontinuous semigroup $T(t)$ $(t \geqslant 0)$ of bounded operators, in $L_p(\Omega)$, satisfying the inequality

$$\|T(t)\| \leqslant Ce^{-\mu t}, \qquad \mu > 0 \tag{4}$$

With the aid of this semigroup one can obtain the generalized solution of the problem (1), (2) as the solution of the nonlinear integral equation

$$u(t, x) = T(t)\varphi(x) + \int_0^t T(t-s) f(s, x, u(s, x))\, ds \tag{5}$$

Theorem. Suppose that for $u(x) \in L_p(\Omega)$, $\|u\| \leqslant \gamma$, $t \geqslant 0$, we have the inequality

$$\|f(t, x, u(x)\| \leqslant k\|u\| + \psi(t)\|u\|^{1+\alpha} \tag{6}$$

where

$$\alpha > 0, \qquad 0 \leqslant k < \frac{\mu}{C}, \quad \int_0^\infty e^{-\alpha(\mu - Ck)t}\psi(t)\, dt < +\infty$$

Then for every ε such that $0 < \varepsilon \leqslant \gamma$, and for

$$\|\varphi\| < \delta = \frac{1}{C}\left[\varepsilon^{-\alpha} + C\alpha \int_0^\infty e^{-\alpha(\mu - Ck)s}\psi(s)\, ds\right]^{-\frac{1}{\alpha}}$$

the solution $u(t, x)$ of the problem (1), (2) will satisfy the inequality

$$\|u(t, x)\| \leqslant \varepsilon e^{-(\mu - Ck)t}$$

that is, the vanishing solution of the equation (1) is exponentially asymptotically stable.

The proof of this is almost identical to the one used in [2].

Note 1. A strongly elliptical operator generates a semigroup with the same properties as the semigroup of a uniformly elliptic operator [3]. Therefore, one can formulate this theorem also for a nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} = -L_i u + f_i(t, x, u_1, \ldots, u_r), \qquad u = (u_1, \ldots, u_r) \quad (i = 1, \ldots, r)$$

where $L = (L_1, \ldots, L_r)$ is a strongly elliptic operator.

Note 2. If Lu is not positive definite, but only bounded from below, and if it does not have pure imaginary points in the spectrum, then one can obtain analogous inequalities for the bounded solutions of equation

(1), exponentially growing estimates from below for the unbounded solutions, and one can establish the asymptotic stability of the vanishing solution in the class of bounded solutions, just as was done in [4].

Note 3. The presence of the constant C in the inequality (4) is important, because this constant can be greater than one for equations of higher order, in contrast to the case $m = 1$ when $C = 1$, as was shown by Sobolevskii [5].

BIBLIOGRAPHY

1. Kostandian, B.A., Ob ustoichivosti resheniia nelineinogo uravneniia teploprovodnosti (On the stability of the solution of a nonlinear equation of heat conduction). *PMM* Vol. 24, No. 6, 1960.

2. Rakhmatullina, L.F., K voprosu ob ustoichivosti resheniia nelineinogo uravneniia teploprovodnosti (On the problem of stability of the solution of the nonlinear heat-conduction equation). *PMM* Vol. 25, No. 3, 1961.

3. Solomiak, M.Z., Analitichnost' polugruppy, porozhdennoi ellipticheskim operatorom v prostranstve L_p (Analiticity of semigroup generated by an elliptic operator in the space L_p). *Dokl. Akad. Nauk SSSR* Vol. 127, No. 1, 1959.

4. Domshlak, Iu.I., K teorii differentsial'nykh uravnenii v banakhovom prostranstve s postoiannym neogranichennym operatorom (On the theory of differential equations in a Banach space with a constant bounded operator). *Dokl. Akad. Nauk AzerbSSR* No. 5, 1962.

5. Sobolevskii, P.E., Ob uravneniiakh parabolicheskogo tipa v banakhovom prostranstve (On equations of the parabolic type in a Banach space). *Tr. Mosk. matem. ob-va* Vol. 10, 1961.

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